q-DEFORMED CLIFFORD ALGEBRA AND LEVEL ZERO FUNDAMENTAL REPRESENTATIONS OF QUANTUM AFFINE ALGEBRAS

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ABSTRACT. We give a realization of the level zero fundamental weight representation $W(\varpi_k)$ of the quantum affine algebra $U_q'(\mathfrak{g})$, when $\mathfrak{g}$ has a maximal parabolic subalgebra of type C_n. We define a semisimple $U_q'(\mathfrak{g})$-module structure on $\Lambda(V)^{\otimes 2}$ in terms of q-deformed Clifford generators, where $\Lambda(V)$ is the exterior algebra generated by a dual natural representation V of $U_q(\mathfrak{sl}_n)$. We show that each $W(\varpi_k)$ appears as an irreducible summand (not necessarily multiplicity free) in $\Lambda(V)^{\otimes 2}$. As a byproduct, we obtain a simple description of the affine crystal structure of $W(\varpi_k)$ in terms of $n \times 2$ binary matrices and their $(\mathfrak{sl}_n, \mathfrak{sl}_2)$-bicrystal structure.

1. INTRODUCTION

Let $\mathfrak{g}$ be an affine Kac-Moody algebra, and $U_q'(\mathfrak{g})$ the associated quantum affine algebra without derivation. For a level zero fundamental weight ϖ_k, Kashiwara introduced a finite dimensional irreducible $U_q'(\mathfrak{g})$-module $W(\varpi_k)$, which is called a level zero fundamental weight module. It is obtained from a level zero extremal weight module $V(\varpi_k)$ by specializing its $U_q'(\mathfrak{g})$-linear automorphism z_k as 1, and has a crystal base and global crystal base [13]. By the works of Chari and Pressley [3, 4], any finite dimensional irreducible $U_q'(\mathfrak{g})$-module is isomorphic to a subquotient of $W(\varpi_{k_1})_{a_1} \otimes \cdots \otimes W(\varpi_{k_r})_{a_r}$ for some $(k_1, a_1), \ldots, (k_r, a_r)$, where $W(\varpi_k)_a$ is obtained by specializing z_k as a. We also refer the reader to [6, 9, 11, 15] for previously known constructions of $W(\varpi_k)$ of various types.

The aim of this article is to introduce a realization of $W(\varpi_k)$ and its crystal for a special class of affine Kac-Moody algebra $\mathfrak{g}$, which has a maximal parabolic subalgebra of type C_n, that is, $\mathfrak{g} = C_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n}^{(2)\dagger}$ and $A_{2n-1}^{(2)}$. Instead of using q-wedge relations for classical Lie algebras of type B, C and D, which were derived via R-matrix by Jing, Misra and Okado [9], we construct a semisimple $U_q'(\mathfrak{g})$-module using a homomorphic image of $U_q'(\mathfrak{g})$ in a q-deformed Clifford algebra, which has

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a simple description of crystal structure and contains all $W(\varpi_k)$ as its irreducible summand.

More precisely, we consider an exterior algebra $\Lambda(V)$ generated by a dual natural representation V of $U_q(\mathfrak{sl}_n) \subset U_q'(\mathfrak{g})$. Based on an action of q-deformed Clifford algebra on $\Lambda(V)$ due to Hayashi [7], we extend the $U_q(\mathfrak{sl}_n)$-action on $\Lambda(V)^{\otimes 2}$ to that of $U_q'(\mathfrak{g})$, and show that $\Lambda(V)^{\otimes 2}$ is a semisimple $U_q'(\mathfrak{g})$-module with a polarizable crystal base. The crystal of $\Lambda(V)^{\otimes 2}$ can be identified with the set of $n \times 2$ binary matrices, whose $U_q'(\mathfrak{g})$-crystal structure has a very simple description (see Figures 1-3). Using its decomposition into connected components, we show that an irreducible summand in $\Lambda(V)^{\otimes 2}$ is generated by an extremal weight vector and then isomorphic to $W(\varpi_k)$ or $W(\varpi_k)_{-1}$ for some k (Theorem 5.4). We also obtain an explicit decomposition of $\Lambda(V)^{\otimes 2}$, where each $W(\varpi_k)$ appears at least once but not necessarily multiplicity free (Corollary 5.6).

Moreover, from a q-deformed skew Howe duality on $\Lambda(\mathbb{C}^n \otimes \mathbb{C}^2)$ (cf. [2, 17] and [5, 18] for its crystal version), we observe that there are additional $U_q(\mathfrak{sl}_2)$-crystal operators $\widetilde{E}$ and $\widetilde{F}$ acting on the crystal of $\Lambda(V)^{\otimes 2}$, which commute with those of $U_q(\mathfrak{sl}_n) \subset U_q'(\mathfrak{g})$. This together with the author's previous work on classical crystals [16] enables us to characterize the crystal of $W(\varpi_k)$ explicitly in terms of binary matrices and their statistics coming from an $\mathfrak{sl}_2$-string with respect to $\widetilde{E}$ and $\widetilde{F}$ (Theorem 5.8). This $(\mathfrak{sl}_n, \mathfrak{sl}_2)$-bicrystal structure also plays a crucial role in the decomposition of $\Lambda(V)^{\otimes 2}$ when $\mathfrak{g}$ is of type $A_{2n-1}^{(2)}$.

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2. Background

Let us briefly recall necessary background for quantum affine algebras and crystal bases (see [13] for more details and references therein).

2.1. **Notations.** Let $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan matrix of affine type with an index set $I = \{\, 0, 1, \ldots, n \,\}$ and let $\mathfrak{g}$ denote the associated affine Kac-Moody algebra with the Cartan subalgebra $\mathfrak{h}$ [10, §4.8]. Let $\{\, \alpha_i \,|\, i \in I \,\} \subset \mathfrak{h}^*$ and $\{\, h_i \,|\, i \in I \,\} \subset \mathfrak{h}$ be the set of simple roots and simple coroots of $\mathfrak{g}$, respectively, with $\langle h_i, \alpha_j \rangle = a_{ij}$. We assume that $\{\, \alpha_i \,|\, i \in I\,\}$ and $\{\, h_i \,|\, i \in I\,\}$ are linearly independent. For $r \in \{0, n\}$, put $I_r = I \setminus \{r\}$ and let $\mathfrak{g}_r$ be the subalgebra of $\mathfrak{g}$ associated to $(a_{ij})_{i,j \in I_r}$.

Let $c = \sum_{i \in I} a_i^\vee h_i$ be the canonical central element and $\delta = \sum_{i \in I} a_i \alpha_i$ the generator of the null roots. Let Λ_i be the fundamental weight such that $\langle h_j, \Lambda_i \rangle = \delta_{ij}$ for $i, j \in I$. We take a weight lattice P such that $\alpha_i, \Lambda_i \in P$ and $h_i \in P^* := \mathrm{Hom}(P, \mathbb{Z})$.

Let $(\ ,\ )$ be a non-degenerate symmetric bilinear form on $\mathfrak{h}^*$ satisfying $\langle h_i, \lambda\rangle = 2(\alpha_i,\lambda)/(\alpha_i,\alpha_i)$ for $i \in I$ and $\lambda \in \mathfrak{h}^*$ and normalized by $(\delta,\lambda) = \langle c, \lambda\rangle$ for $\lambda \in P$. Note that $(\alpha_i,\alpha_j) = a_i^\vee a_i^{-1} a_{ij}$ for $i, j \in I$.

Let $\mathfrak{h}^*_{\mathrm{cl}} = \mathfrak{h}^*/\mathbb{Q}\delta$ and $\mathrm{cl} : \mathfrak{h}^* \longrightarrow \mathfrak{h}^*_{\mathrm{cl}}$ the projection. Let $\mathfrak{h}^{*0} = \{\, \lambda \in \mathfrak{h}^* \mid \langle c, \lambda\rangle = 0 \,\}$ and $\mathfrak{h}^{*0}_{\mathrm{cl}} = \mathrm{cl}(\mathfrak{h}^{*0})$. Let $P_{\mathrm{cl}} = \mathrm{cl}(P)$, $P^0 = \mathfrak{h}^{*0} \cap P$, and $P^0_{\mathrm{cl}} = \mathrm{cl}(P^0)$.

For $i \in I$, let s_i be the simple reflection in $GL(\mathfrak{h}^*)$ given by $s_i(\lambda) = \lambda - \langle h_i, \lambda\rangle \alpha_i$. Let W be the Weyl group of $\mathfrak{g}$ generated by s_i for $i \in I$. Note that W naturally induces an action on $\mathfrak{h}^{*0}_{\mathrm{cl}}$, whose image we denote by W_{cl}, and W_{cl} is generated by s_i for $i \in I_0$.

2.2. **Quantum affine algebra and crystal base.** Let d be the smallest positive integer such that $(\alpha_i,\alpha_i)/2 \in \frac{1}{d}\mathbb{Z}$ for $i \in I$. Let q be an indeterminate and put $q_s = q^{1/d}$. Let $K = \mathbb{Q}(q_s)$. The quantum affine algebra $U_q(\mathfrak{g})$ is the unital associative K-algebra generated by e_i, f_i and q^h for $i \in I$ and $h \in \frac{1}{d}P^*$ subject to the relations:

$$q^0 = 1, \quad q^{h+h'} = q^h q^{h'},$$
$$q^h e_i = q^{\langle h,\alpha_i\rangle} e_i q^h, \quad q^h f_i = q^{-\langle h,\alpha_i\rangle} f_i q^h,$$
$$e_i f_j - f_j e_i = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(k)} e_j e_i^{(1-a_{ij}-k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(k)} f_j f_i^{(1-a_{ij}-k)} = 0 \qquad (i \neq j),$$

where $q_i = q^{(\alpha_i,\alpha_i)/2}$, $t_i = q^{(\alpha_i,\alpha_i)h_i/2}$, and

$$[k]_i = \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}}, \quad [k]_i! = \prod_{s=1}^{k} [s]_i, \quad e_i^{(k)} = \frac{1}{[k]_i!} e_i^k,\ f_i^{(k)} = \frac{1}{[k]_i!} f_i^k,$$

for $i \in I$ and $k \geq 0$. Recall $U_q(\mathfrak{g})$ has a comultiplication Δ given by

$$\Delta(q^h) = q^h \otimes q^h,$$
$$\Delta(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i,$$
$$\Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i,$$

for $i \in I$ and $h \in \frac{1}{d}P^*$.

We denote by $U_q'(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i and q^h for $i \in I$ and $h \in \frac{1}{d}(P_{\mathrm{cl}})^*$. Let z be an indeterminate. For a $U_q'(\mathfrak{g})$-module M with weight space decomposition $M = \bigoplus_{\lambda \in P_{\mathrm{cl}}} M_\lambda$, let $M_{\mathrm{aff}} = K[z, z^{-1}] \otimes M$ be a $U_q(\mathfrak{g})$-module, where e_i and $f_i \in U_q'(\mathfrak{g})$ act by $z^{\delta_{0i}} \otimes e_i$ and $z^{-\delta_{0i}} \otimes f_i$, respectively for $i \in I$, and $\mathrm{wt}(z^k \otimes m) = \mathrm{wt}(m) + k\delta$ for $m \in M$ and $k \in \mathbb{Z}$. Here wt denotes the weight function. For $a \in K$, we define a $U_q'(\mathfrak{g})$-module $M_a = M_{\mathrm{aff}}/(z-a)M_{\mathrm{aff}}$.

Let M be an integrable module over $U_q(\mathfrak{g})$ or $U_q'(\mathfrak{g})$ having weight space decomposition $M=\bigoplus_\lambda M_\lambda$ with $\dim M_\lambda<\infty$ for $\lambda\in P$ or $P_{\rm cl}$. For $u\in M_\lambda$ and $i\in I$, we have $u=\sum_{r\geq 0,-\langle h_i,\lambda\rangle} f_i^{(r)}u_r$, where $e_iu_r=0$ for all $r\geq 0$. We define $\widetilde{e}_i$ and $\widetilde{f}_i$ by $\widetilde{e}_iu=\sum_{r\geq 1} f_i^{(r-1)}u_r$ and $\widetilde{f}_iu=\sum_{r\geq 0} f_i^{(r+1)}u_r$. Let $\mathbb{A}$ denote the subring of K consisting of all rational functions which are regular at $q_s=0$. A pair (L,B) is called a crystal base of M if

(1) L is an $\mathbb{A}$-lattice of M, where $L=\bigoplus_\lambda L_\lambda$ with $L_\lambda=L\cap M_\lambda$,
(2) $\widetilde{e}_iL\subset L$ and $\widetilde{f}_iL\subset L$ for $i\in I$,
(3) B is a $\mathbb{Q}$-basis of L/q_sL, where $B=\bigsqcup_\lambda B_\lambda$ with $B_\lambda=B\cap(L/q_sL)_\lambda$,
(4) $\widetilde{e}_iB\subset B\sqcup\{0\}$, $\widetilde{f}_iB\subset B\sqcup\{0\}$ for $i\in I$,
(5) for $b,b'\in B$ and $i\in I$, $\widetilde{f}_ib=b'$ if and only if $b=\widetilde{e}_ib'$.

Following [1] (cf. [11]), we say that a symmetric bilinear form $(\ ,\ )$ on M is a polarization if

$$(xu,v)=(u,\eta(x)v) \tag{2.1}$$

for $x\in U_q(\mathfrak{g})$ or $U_q'(\mathfrak{g})$, $u,v\in M$, where η is the anti-automorphism given by

$$\eta(q^h)=q^h,\quad \eta(e_i)=q_i^{-1}t_i^{-1}f_i,\quad \eta(f_i)=q_i^{-1}t_ie_i\quad (i\in I),$$

and say that a crystal base (L,B) of M is polarizable if $(L,L)\subset\mathbb{A}$ with respect to a polarization on M and B is orthonormal (up to scalar multiplication by ± 1) with respect to the induced $\mathbb{Q}$-bilinear form $(\ ,\ )_0$ on L/q_sL. If $(\ ,\ )_{M_i}$ is a polarization of M_i $(i=1,2)$, then $M_1\otimes M_2$ has a polarization given by $(u_1\otimes u_2,v_1\otimes v_2)_{M_1\otimes M_2}=(u_1,v_1)_{M_1}(u_2,v_2)_{M_2}$ for $u_i,v_i\in M_i$. If (L_i,B_i) is a polarizable crystal base of M_i, then $(L_1\otimes L_2,B_1\otimes B_2)$ is a polarizable crystal base of $M_1\otimes M_2$.

Proposition 2.1 (Theorem 2.12 in [1]). *If M has a polarizable crystal base, then M is completely reducible.*

2.3. **Level zero fundamental weight module.** For a regular crystal B, we define the action of W as follows. For $i\in I$ and $b\in B$,

$$\mathtt{S}_{s_i}(b)=\begin{cases}\widetilde{f}_i^{\langle h_i,{\rm wt}(b)\rangle}b, & \text{if } \langle h_i,{\rm wt}(b)\rangle\geq 0,\\ \widetilde{e}_i^{-\langle h_i,{\rm wt}(b)\rangle}b, & \text{if } \langle h_i,{\rm wt}(b)\rangle\leq 0.\end{cases} \tag{2.2}$$

For $w\in W$ with a reduced expression $w=s_{i_1}\ldots s_{i_r}$, we let $\mathtt{S}_w=\mathtt{S}_{s_{i_1}}\ldots\mathtt{S}_{s_{i_r}}$.

Let u_λ be a weight vector of an integrable module M over $U_q(\mathfrak{g})$ or $U_q'(\mathfrak{g})$ with weight λ. Then u_λ is called an extremal weight vector of extremal weight λ if there exists $\{u_{w\lambda}\}_{w\in W}$ such that

(1) $u_{w\lambda}=u_\lambda$ if w is the identity,

(2) $e_i u_{w\lambda} = 0$ and $f_i^{(\langle h_i,\lambda\rangle)} u_{w\lambda} = u_{s_i w\lambda}$ if $\langle h_i, w\lambda\rangle \geq 0$,
(3) $f_i u_{w\lambda} = 0$ and $e_i^{(-\langle h_i,\lambda\rangle)} u_{w\lambda} = u_{s_i w\lambda}$ if $\langle h_i, w\lambda\rangle \leq 0$.

If u_λ is an extremal weight vector, we denote $u_{w\lambda}$ by $S_w u_\lambda$ for $w \in W$.

For $\lambda \in P$, define $V(\lambda)$ to be a $U_q(\mathfrak{g})$-module generated by a vector u_λ of weight λ subject to the relations such that u_λ is an extremal weight vector. We call $V(\lambda)$ an extremal weight module with extremal weight λ. The notion of extremal weight module was introduced by Kashiwara and it was proved that $V(\lambda)$ has a crystal base $(L(\lambda), B(\lambda))$ and a global crystal base [12]. Note that $\mathtt{S}_w u_\lambda \equiv S_w u_\lambda \pmod{qL(\lambda)}$ for $w \in W$.

For $i \in I_0$, let

$$\varpi_i = \Lambda_i - \langle c, \Lambda_i\rangle \Lambda_0 = \Lambda_i - a_i^\vee \Lambda_0 \in P^0, \tag{2.3}$$

which is called the level zero fundamental weight. We have $\{\, k \in \mathbb{Z} \,|\, \varpi_i + k\delta \in W\varpi_i \,\} = \mathbb{Z}d_i$, where $d_i = \max\{1, (\alpha_i, \alpha_i)/2\}$ except in the case $d_i = 1$ when $\mathfrak{g} = A_{2n}^{(2)}$ and $i = n$. There exists a $U_q'(\mathfrak{g})$-linear automorphism z_i on $V(\varpi_i)$ of weight $d_i\delta$ sending u_{ϖ_i} to $u_{\varpi_i + d_i\delta}$. We define

$$W(\varpi_i) = V(\varpi_i)/(z_i - 1)V(\varpi_i), \tag{2.4}$$

which is called a level zero fundamental representation of $U_q'(\mathfrak{g})$ [13]. They play a crucial role, especially as building blocks of finite dimensional $U_q'(\mathfrak{g})$-modules. The following properties of $W(\varpi_i)$ are known, which is a part of [13, Theorem 5.17].

Theorem 2.2.

(1) $W(\varpi_i)$ *is a finite dimensional irreducible integrable* $U_q'(\mathfrak{g})$*-module.*
(2) $W(\varpi_i)$ *has a global crystal base with a simple crystal.*
(3) $\dim W(\varpi_i)_\mu = 1$ *for* $\mu \in W\mathrm{cl}(\varpi_i)$.
(4) *The weight of an extremal weight vector of* $W(\varpi_i)$ *is in* $W\mathrm{cl}(\varpi)$.
(5) *The set of weights of* $W(\varpi_i)$ *is the intersection of* $\mathrm{cl}(\varpi_i + \sum_{i\in I}\mathbb{Z}\alpha_i)$ *and the convex hull of* $W\mathrm{cl}(\varpi_i)$.
(6) *Any finite dimensional irreducible integrable* $U_q'(\mathfrak{g})$*-module with* $\mathrm{cl}(\varpi_i)$ *as an extremal weight is isomorphic to* $W(\varpi_i)_a$ *for some* $a \in K \setminus \{0\}$.

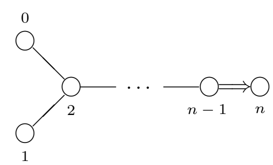
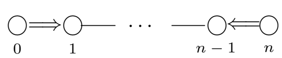
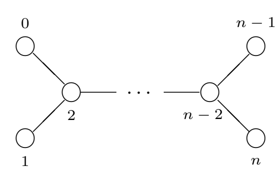
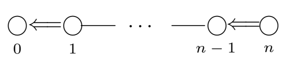
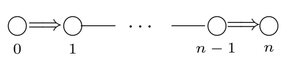
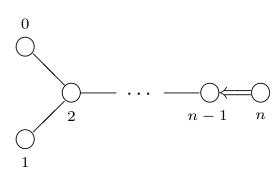
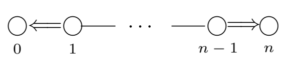
3. q-wedge spaces and level zero representations

3.1. **Non-exceptional affine algebras.** Throughout this paper, we assume that $\mathfrak{g}$ is of type $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n-1}^{(2)}$, $D_{n+1}^{(2)}$ (called non-exceptional affine type together with $A_n^{(1)}$) following [10] for the labeling of simple roots. Note that $\mathfrak{g}_0 \cap \mathfrak{g}_n = A_{n-1}$ and $\mathfrak{g}_r$ $(r = 0, n)$ is one of B_n, C_n and D_n. Let us denote the type of $\mathfrak{g}_r$ by a

partition or a Young diagram $\diamondsuit$ as follows:

$$\diamondsuit = \begin{cases} \square = (1) & \text{for } B_n, \\ \square\square = (2) & \text{for } C_n, \\ \begin{smallmatrix}\square\\\square\end{smallmatrix} = (1,1) & \text{for } D_n. \end{cases}$$

Since the type of $\mathfrak{g}$ is completely determined by those of $\mathfrak{g}_r$ ($r = 0, n$), we may identify the type of $\mathfrak{g}$ with a pair of partitions $\boldsymbol{\diamondsuit} = (\diamondsuit_0, \diamondsuit_n)$, where $\diamondsuit_r$ is the type of $\mathfrak{g}_{n-r}$. Since $\mathfrak{g}_0 \cap \mathfrak{g}_n = A_{n-1}$ is fixed, we may understand that $\diamondsuit_r$ is determined by α_r for $r \in \{0, n\}$ in the Dynkin diagram of $\mathfrak{g}$. This convention will be useful when we realize $W(\varpi_i)$ and its crystal in later sections. For the readers' convenience, we list the diagrams of $\mathfrak{g}$ and the associated pair $\boldsymbol{\diamondsuit}$.

$B_n^{(1)}$		$(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square)$
$C_n^{(1)}$		$(\square\square, \square\square)$
$D_n^{(1)}$		$(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \begin{smallmatrix}\square\\\square\end{smallmatrix})$
$A_{2n}^{(2)}$		$(\square, \square\square)$
$A_{2n}^{(2)\dagger}$		$(\square\square, \square)$
$A_{2n-1}^{(2)}$		$(\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square)$
$D_{n+1}^{(2)}$		$(\square, \square)$

Here $A_{2n}^{(2)\dagger}$ is a different labeling of simple roots for $A_{2n}^{(2)}$, and in this case, we have

$$\varpi_n = 2\Lambda_n - \Lambda_0, \quad \varpi_i = \Lambda_i - \Lambda_0 \quad (i = 1, \ldots, n-1). \tag{3.1}$$

3.2. **q-deformed Cliffors algebra.** Let $[\overline{n}]=\{\overline{n}<\ldots<\overline{1}\}$ be a linearly ordered set. Consider a q-deformed Clifford algebra $\mathscr{A}_q=\mathscr{A}_q(n)$ [7], which is an associative K-algebra with 1 generated by ψ_a, ψ_a^*, ω_a and ω_a^{-1} for $a\in[\overline{n}]$ subject to the following relations:

$$\begin{gathered}
\omega_a\omega_b=\omega_b\omega_a, \quad \omega_a\omega_a^{-1}=1,\\
\omega_a\psi_b\omega_a^{-1}=q^{\delta_{ab}}\psi_b, \qquad \omega_a\psi_b^*\omega_a^{-1}=q^{-\delta_{ab}}\psi_b^*\\
\psi_a\psi_b+\psi_b\psi_a=0, \qquad \psi_a^*\psi_b^*+\psi_b^*\psi_a^*=0,\\
\psi_a\psi_b^*+\psi_b^*\psi_a=0 \qquad (a\neq b),\\
\psi_a\psi_a^*=\frac{q\omega_a-q^{-1}\omega_a^{-1}}{q-q^{-1}}, \quad \psi_a^*\psi_a=-\frac{\omega_a-\omega_a^{-1}}{q-q^{-1}}.
\end{gathered}$$

Let $\mathscr{E}_q$ be the left $\mathscr{A}_q$-module generated by $|0\rangle$ satisfying $\psi_a^*|0\rangle=0$ and $\omega_a|0\rangle=q^{-1}|0\rangle$ for $a\in[\overline{n}]$. Then $\mathscr{E}_q$ is an irreducible $\mathscr{A}_q$-module with a K-linear basis $\{\,\psi_{\mathbf{m}}|0\rangle\,|\,\mathbf{m}\in\mathbf{B}\,\}$ (cf. [7, Proposition 2.1]), where $\mathbf{B}=\{\,(m_a)\,|\,a\in[\overline{n}],\ m_a\in\mathbb{Z}_2\,\}$, and $\psi_{\mathbf{m}}|0\rangle=\psi_{\overline{n}}^{m_{\overline{n}}}\cdots\psi_{\overline{1}}^{m_{\overline{1}}}|0\rangle$ for $\mathbf{m}=[m_a]\in\mathbf{B}$.

We put

$$\Lambda(V)=\mathscr{E}_{q_1}, \tag{3.2}$$

where $q_1=q^{(\alpha_1,\alpha_1)/2}$ is equal to $q^{1/2}$ for $C_n^{(1)}$, q^2 for $D_{n+1}^{(2)}$, and q otherwise. One may regard $\Lambda(V)$ as an exterior algebra generated by an n-dimensional space V with basis $\{v_{\overline{n}},\ldots,v_{\overline{1}}\}$ by identifying $\psi_{i_1}\ldots\psi_{i_k}|0\rangle$ with $v_{i_1}\wedge\ldots\wedge v_{i_k}$ for $\overline{n}\leq i_1<\cdots<i_k\leq\overline{1}$. Here, we understand V as the dual natural representation of $U_q(\mathfrak{sl}_n)\subset U_q'(\mathfrak{g})$. Then

Proposition 3.1 (Theorem 3.2 in [7])**.** *$\Lambda(V)$ has a $U_q(\mathfrak{sl}_n)$-module structure, where*

$$t_i\longmapsto\omega_{\overline{i+1}}\omega_{\overline{i}}^{-1}, \quad e_i\longmapsto\psi_{\overline{i+1}}\psi_{\overline{i}}^*, \quad f_i\longmapsto\psi_{\overline{i+1}}\psi_{\overline{i}}^*,$$

for $i=1,\ldots,n-1$

Let $(\ ,\ )_{\Lambda(V)}$ be a non-degenerate symmetric bilinear form on $\Lambda(V)$ such that

$$(\psi_{\mathbf{m}}|0\rangle,\psi_{\mathbf{m}'}|0\rangle)_{\Lambda(V)}=\delta_{\mathbf{m}\,\mathbf{m}'}, \tag{3.3}$$

for $\mathbf{m},\mathbf{m}'\in\mathbf{B}$. Then it is straightforward to check that $\Lambda(V)$ has a polarizable crystal base $(L(\Lambda(V)),B(\Lambda(V)))$ with respect to $(\ ,\ )_{\Lambda(V)}$, where

$$L(\Lambda(V))=\sum_{\mathbf{m}\in\mathbf{B}}\mathbb{A}\,\psi_{\mathbf{m}}|0\rangle, \quad B(\Lambda(V))=\{\,\psi_{\mathbf{m}}|0\rangle\ \ (\mathrm{mod}\ qL(\Lambda(V)))\,|\,\mathbf{m}\in\mathbf{B}\,\}.$$

We may identify $B(\Lambda(V))$ with $\mathbf{B}$, and we have for $i=1,\dots,n-1$ and $\mathbf{m}=(m_a)\in\mathbf{B}$,

$$
\begin{aligned}
\widetilde{e}_i\mathbf{m} &= \begin{cases} \mathbf{m}+\mathbf{e}_{\overline{i+1}}-\mathbf{e}_{\overline{i}}, & \text{if } (m_{\overline{i+1}}, m_{\overline{i}})=(0,1),\\ 0, & \text{otherwise,}\end{cases}\\
\widetilde{f}_i\mathbf{m} &= \begin{cases} \mathbf{m}-\mathbf{e}_{\overline{i+1}}+\mathbf{e}_{\overline{i}}, & \text{if } (m_{\overline{i+1}}, m_{\overline{i}})=(1,0),\\ 0, & \text{otherwise,}\end{cases}
\end{aligned}
\tag{3.4}
$$

where $\mathbf{e}_a\in\mathbf{M}$ corresponds to $\psi_a|0\rangle$ with 1 at the a-th component and 0 elsewhere, for $a\in[\overline{n}]$.

3.3. **$U_q'(\mathfrak{g})$-module structure on $\Lambda(V)^{\otimes 2}$.** Now, we will construct a $U_q'(\mathfrak{g})$-module structure on $\Lambda(V)$ or $\Lambda(V)^{\otimes 2}$ by extending the action of $U_q(A_{n-1})$.

Proposition 3.2. *Suppose that $\diamondsuit_r=\square$ or $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ for some $r\in\{0,n\}$. Then $\Lambda(V)$ has a $U_q(\mathfrak{g}_{n-r})$-module structure, where*

$$
\begin{cases}
t_0\longmapsto q_0\omega_{\overline{1}}, & e_0\longmapsto \psi_{\overline{1}}, \quad f_0\longmapsto \psi_{\overline{1}}^*, \quad \text{if } \diamondsuit_0=\square,\\
t_0\longmapsto q_0\omega_{\overline{1}}\omega_{\overline{2}}, & e_0\longmapsto \psi_{\overline{1}}\psi_{\overline{2}}, \quad f_0\longmapsto \psi_{\overline{2}}^*\psi_{\overline{1}}^*, \quad \text{if } \diamondsuit_0=\begin{smallmatrix}\square\\\square\end{smallmatrix},
\end{cases}
$$

$$
\begin{cases}
t_n\longmapsto q_n^{-1}\omega_{\overline{n}}^{-1}, & e_n\longmapsto \psi_{\overline{n}}^*, \quad f_n\longmapsto \psi_{\overline{n}}, \quad \text{if } \diamondsuit_n=\square,\\
t_n\longmapsto q_n^{-1}(\omega_{\overline{n}}\omega_{\overline{n-1}})^{-1}, & e_n\longmapsto \psi_{\overline{n}}^*\psi_{\overline{n-1}}^*, \quad f_n\longmapsto \psi_{\overline{n-1}}\psi_{\overline{n}}, \quad \text{if } \diamondsuit_n=\begin{smallmatrix}\square\\\square\end{smallmatrix},
\end{cases}
$$

and $(L(\Lambda(V)),B(\Lambda(V)))$ is a polarizable crystal base of $\Lambda(V)$ as a $U_q(\mathfrak{g}_{n-r})$-module with respect to $(\ ,\ )_{\Lambda(V)}$.

Proof. Suppose that $r=n$. Then $\Lambda(V)$ is a $U_q(\mathfrak{g}_0)$-module by [7, Theorem 4.1] with a little modification (cf. [16, Proposition 5.3] on which our presentation is based on), and $(L(\Lambda(V)),B(\Lambda(V)))$ is its crystal base as a $U_q(\mathfrak{g}_0)$-module by [16, Theorem 5.6]. It is also easy to check that $(L(\Lambda(V)),B(\Lambda(V)))$ is polarizable. The proof for $r=0$ is almost the same. $\square$

Under the hypothesis of Proposition 3.2, we have for $\mathbf{m}=(m_a)\in\mathbf{B}$

$$
\widetilde{e}_r\mathbf{m}=\begin{cases}
\mathbf{m}-\mathbf{e}_{\overline{n}}, & \text{if } r=n \text{ with } \diamondsuit_n=\square \text{ and } m_{\overline{n}}=1,\\
\mathbf{m}+\mathbf{e}_{\overline{1}}, & \text{if } r=0 \text{ with } \diamondsuit_0=\square \text{ and } m_{\overline{1}}=0,\\
\mathbf{m}-\mathbf{e}_{\overline{n}}-\mathbf{e}_{\overline{n-1}}, & \text{if } r=n \text{ with } \diamondsuit_n=\begin{smallmatrix}\square\\\square\end{smallmatrix} \text{ and } m_{\overline{n}}=m_{\overline{n-1}}=1,\\
\mathbf{m}+\mathbf{e}_{\overline{2}}+\mathbf{e}_{\overline{1}}, & \text{if } r=0 \text{ with } \diamondsuit_0=\begin{smallmatrix}\square\\\square\end{smallmatrix} \text{ and } m_{\overline{2}}=m_{\overline{1}}=0,\\
0, & \text{otherwise.}
\end{cases}
\tag{3.5}
$$

Note that $\widetilde{f}_r \mathbf{m} = \mathbf{m}'$ is determined by the relation $\widetilde{e}_r \mathbf{m}' = \mathbf{m}$ if $\widetilde{f}_r \mathbf{m} \neq 0$. In fact, $\mathbf{B}$ is the crystal of the spin representation (resp. the sum of two spin representations) when $\mathfrak{g}_r = B_n$ (resp. D_n) [14].

Proposition 3.3. *Suppose that $\diamondsuit_r = \square\!\square$ for some $r \in \{0, n\}$. Then $\Lambda(V)^{\otimes 2}$ has a $U_q(\mathfrak{g}_{n-r})$-module structure, where*

$$\begin{cases} t_0 \longmapsto q_0 \omega_{\overline{1}} \otimes \omega_{\overline{1}}, & e_0 \longmapsto \psi_{\overline{1}} \otimes \psi_{\overline{1}}, \qquad f_0 \longmapsto \psi^*_{\overline{1}} \otimes \psi^*_{\overline{1}}, \\ t_n \longmapsto q_n^{-1} \omega_{\overline{n}}^{-1} \otimes \omega_{\overline{n}}^{-1}, & e_n \longmapsto \psi^*_{\overline{n}} \otimes \psi^*_{\overline{n}}, \qquad f_n \longmapsto \psi_{\overline{n}} \otimes \psi_{\overline{n}}, \end{cases}$$

and $(L(\Lambda(V))^{\otimes 2}, B(\Lambda(V))^{\otimes 2})$ is a polarizable crystal base of $\Lambda(V)^{\otimes 2}$ as a $U_q(\mathfrak{g}_r)$-module with respect to $(\ ,\ )_{\Lambda(V)^{\otimes 2}}$, which is induced from $(\ ,\ )_{\Lambda(V)}$.

Proof. The proof is similar to that of Proposition 3.2. $\square$

Under the hypothesis of Proposition 3.3, we have for $\mathbf{m} \otimes \mathbf{m}' = (m_a) \otimes (m'_a) \in \mathbf{B}^{\otimes 2}$

$$\widetilde{e}_r (\mathbf{m} \otimes \mathbf{m}') = \begin{cases} (\mathbf{m} - \mathbf{e}_{\overline{n}}) \otimes (\mathbf{m}' - \mathbf{e}_{\overline{n}}), & \text{if } r = n \text{ and } m_{\overline{n}} = m'_{\overline{n}} = 1, \\ (\mathbf{m} + \mathbf{e}_{\overline{1}}) \otimes (\mathbf{m}' + \mathbf{e}_{\overline{1}}), & \text{if } r = 0 \text{ and } m_{\overline{1}} = m'_{\overline{1}} = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.6}$$

Now, we have the following.

Proposition 3.4. *Let $\mathfrak{g}$ be an affine Kac-Moody algebra of type $\diamondsuit = (\diamondsuit_0, \diamondsuit_n)$.*

(1) *If $\diamondsuit_0, \diamondsuit_n \neq \square\!\square$, then $\Lambda(V)$ is a finite dimensional semisimple $U'_q(\mathfrak{g})$-module with a polarizable crystal base $(L(\Lambda(V)), B(\Lambda(V)))$ with $\mathrm{wt}(|0\rangle) = \mathrm{cl}(\varpi_n)$.*

(2) *$\Lambda(V)^{\otimes 2}$ is a finite dimensional semisimple $U'_q(\mathfrak{g})$-module with a polarizable crystal base $(L(\Lambda(V))^{\otimes 2}, B(\Lambda(V))^{\otimes 2})$ with*

$$\mathrm{wt}(|0\rangle \otimes |0\rangle) = \begin{cases} 2\mathrm{cl}(\varpi_n), & \textit{if } \diamondsuit_0, \diamondsuit_n \neq \square\!\square\ , \\ \mathrm{cl}(\varpi_n), & \textit{if } \diamondsuit_0 \textit{ or } \diamondsuit_n = \square\!\square\ . \end{cases}$$

Proof. It follows from Propositions 3.2 and 3.3 that $\Lambda(V)^{\otimes N}$ $(N = 1, 2)$ is a $U'_q(\mathfrak{g})$-module, and $(L(\Lambda(V))^{\otimes N}, B(\Lambda(V))^{\otimes N})$ is its polarizable crystal base, which also implies that $\Lambda(V)^{\otimes N}$ is semisimple by Proposition 2.1. $\square$

3.4. **Binary matrices and crystal of $\Lambda(V)^{\otimes 2}$.** Let $\mathbf{M}$ be the set of binary matrices $\mathbf{m} = [m_{ab}]$ $(a \in [\overline{n}], b \in \{1, 2\})$. Let $\mathbf{m}_{(a)} = [m_{a1}\, m_{a2}]$ be the a-th row and $\mathbf{m}^{(b)} = [m_{ab}]$ the b-th column of $\mathbf{m}$ for $a \in [\overline{n}]$ and $b = 1, 2$. By Theorem 3.4 (2), we may regard $\mathbf{M}$ as a crystal of $\Lambda(V)^{\otimes 2}$ identifying $\mathbf{m} \in \mathbf{M}$ with $\psi_{\mathbf{m}^{(1)}}|0\rangle \otimes \psi_{\mathbf{m}^{(2)}}|0\rangle \in B(\Lambda(V))^{\otimes 2} = \mathbf{B}^{\otimes 2}$.

Let us describe $\widetilde{e}_i$ for $i \in \{0, n\}$ on $B(\Lambda(V))^{\otimes 2}$ explicitly in terms of $\mathbf{M}$ using (3.5), (3.6) and tensor product rule of crystals (see Figures 1 and 2, for example). This will be useful for the arguments in the next section.

Case 1. $\diamondsuit_i = \square$.

Since $\widetilde{e}_n \mathbf{m}$ (resp. $\widetilde{e}_0 \mathbf{m}$) depend only on $\mathbf{m}_{(\overline{n})}$ (resp. $\mathbf{m}_{(\overline{1})}$), it is enough to describe in terms of $\mathbf{m}_{(\overline{n})}$ and $\mathbf{m}_{(\overline{1})}$. We have

$$\widetilde{e}_n \mathbf{m}_{(\overline{n})} = \begin{cases} [0\ 0], & \text{if } \mathbf{m}_{(\overline{n})} = [1\ 0], \\ [1\ 0], & \text{if } \mathbf{m}_{(\overline{n})} = [1\ 1], \\ 0, & \text{otherwise}, \end{cases} \quad \widetilde{e}_0 \mathbf{m}_{(\overline{1})} = \begin{cases} [0\ 1], & \text{if } \mathbf{m}_{(\overline{1})} = [0\ 0], \\ [1\ 1], & \text{if } \mathbf{m}_{(\overline{1})} = [0\ 1], \\ 0, & \text{otherwise}. \end{cases}$$

Case 2. Suppose that $\diamondsuit_i = \square\square$.

As in Case 1, $\widetilde{e}_n \mathbf{m}$ (resp. $\widetilde{e}_0 \mathbf{m}$) depend only on $\mathbf{m}_{(\overline{n})}$ (resp. $\mathbf{m}_{(\overline{1})}$). We have

$$\widetilde{e}_n \mathbf{m}_{(\overline{n})} = \begin{cases} [0\ 0], & \text{if } \mathbf{m}_{(\overline{n})} = [1\ 1], \\ 0, & \text{otherwise}, \end{cases} \quad \widetilde{e}_0 \mathbf{m}_{(\overline{1})} = \begin{cases} [1\ 1], & \text{if } \mathbf{m}_{(\overline{1})} = [0\ 0], \\ 0, & \text{otherwise}. \end{cases}$$

Case 3. Suppose that $\diamondsuit_i = \begin{array}{c}\square\\[-4pt]\square\end{array}$.

It is enough to describe in terms of the 2×2-submatrix $\begin{bmatrix} \mathbf{m}_{(\overline{2})} \\ \mathbf{m}_{(\overline{1})} \end{bmatrix}$ or $\begin{bmatrix} \mathbf{m}_{(\overline{n})} \\ \mathbf{m}_{(\overline{n-1})} \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$. We have for $i = n$

$$\widetilde{e}_n \begin{bmatrix} \mathbf{m}_{(\overline{n})} \\ \mathbf{m}_{(\overline{n-1})} \end{bmatrix} = \begin{cases} \begin{bmatrix} p & 0 \\ r & 0 \end{bmatrix}, & \text{if } \begin{bmatrix} p \\ r \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} q \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\[2ex] \begin{bmatrix} 0 & q \\ 0 & s \end{bmatrix}, & \text{if } \begin{bmatrix} p \\ r \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} q \\ s \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\[2ex] 0, & \text{otherwise}, \end{cases}$$

and for $i = 0$

$$\widetilde{e}_0 \begin{bmatrix} \mathbf{m}_{(\overline{2})} \\ \mathbf{m}_{(\overline{1})} \end{bmatrix} = \begin{cases} \begin{bmatrix} p & 1 \\ r & 1 \end{bmatrix}, & \text{if } \begin{bmatrix} p \\ r \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} q \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\[2ex] \begin{bmatrix} 1 & q \\ 1 & s \end{bmatrix}, & \text{if } \begin{bmatrix} p \\ r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} q \\ s \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\[2ex] 0, & \text{otherwise}. \end{cases}$$

4. Crystal structure on $\Lambda(V)^{\otimes 2}$

4.1. **Decomposition of the crystal of $\Lambda(V)^{\otimes 2}$.** Suppose that $\mathfrak{g}$ is of type $\diamondsuit = (\diamondsuit_0, \diamondsuit_n)$ with $\diamondsuit_0, \diamondsuit_n \neq \square\square$. Let $\mathbf{v}_n = (0, \ldots, 0)$ and $\mathbf{v}_{n-1} = \mathbf{e}_{\overline{n}}$. Then it is not difficult to see that

$$\mathbf{B} = \begin{cases} C(\mathbf{v}_n), & \text{if } \diamondsuit = (\square, \square), \\ C(\mathbf{v}_n) \sqcup C(\mathbf{v}_{n-1}), & \text{otherwise}, \end{cases} \tag{4.1}$$

where $C(\mathbf{v})$ denotes the connected component of $\mathbf{v}$ in $\mathbf{B}$ with respect to $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I$.

Next, suppose that $\mathfrak{g}$ is of type $\diamondsuit = (\diamondsuit_0, \diamondsuit_n)$ with $\diamondsuit_0$ or $\diamondsuit_n = \square\square$. For $0 \leq k \leq n$ and $0 \leq l \leq n-k$, let $\mathbf{v}_{k,l} = \left[\mathbf{v}_{k,l}^{(1)} \ \mathbf{v}_{k,l}^{(2)} \right] \in \mathbf{M}$ be given by

$$\mathbf{v}_{k,l}^{(1)} = \mathbf{e}_{\overline{n}} + \cdots + \mathbf{e}_{\overline{n-l+1}}, \quad \mathbf{v}_{k,l}^{(2)} = \mathbf{e}_{\overline{n-l}} + \cdots + \mathbf{e}_{\overline{k+1}},$$

where we understand $\mathbf{e}_i$ as a column vector and $\mathbf{v}_{k,l}^{(1)}$ (resp. $\mathbf{v}_{k,l}^{(2)}$) as a zero vector when $l = 0$ (resp. $l = n-k$). Note that the number of 1's in $\mathbf{v}_{k,l}$ is $n-k$, while the number of 1's in the first column is l. We have $\widetilde{e}_i \mathbf{v}_{k,l} = 0$ for all $i \in I \setminus \{0, n\}$ by (3.4) and tensor product rule of crystals, where

$$\mathrm{wt}(\mathbf{v}_{k,l}) = \begin{cases} \mathrm{cl}(\varpi_k), & \text{if } 1 \leq k \leq n, \\ 0, & \text{if } k = 0. \end{cases} \tag{4.2}$$

For $\mathbf{m} \in \mathbf{M}$, let $C(\mathbf{m})$ be the connected component of $\mathbf{m}$ in $\mathbf{M}$ with respect to $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I$. Now using Section 3.4, we have the following decomposition of $\mathbf{M}$.

Proposition 4.1. *Suppose that $\mathfrak{g}$ is of type $\diamondsuit = (\diamondsuit_0, \diamondsuit_n)$ with $\diamondsuit_0$ or $\diamondsuit_n = \square\square$. Then as a $U_q'(\mathfrak{g})$-crystal,*

$$\mathbf{M} = \bigsqcup_{(k,l) \in H^{\diamondsuit}} C(\mathbf{v}_{k,l}),$$

where

$$H^{\diamondsuit} = \begin{cases} \{\, (k,l) \,|\, 0 \leq k \leq n,\ 0 \leq l \leq n-k \,\}, & \text{if } \ \diamondsuit = (\square\square, \square\square), \\ \{\, (k, n-k) \,|\, 0 \leq k \leq n \,\}, & \text{if } \ \diamondsuit = (\square, \square\square), \\ \{\, (k, 0) \,|\, 0 \leq k \leq n \,\}, & \text{if } \ \diamondsuit = (\square\square, \square), \\ \{\, (k, n-k) \,|\, 0 \leq k \leq n \,\} \cup \{(0, n-1)\}, & \text{if } \ \diamondsuit = (\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square). \end{cases}$$

Proof. Let C be a connected component in $\mathbf{M}$. Choose $\mathbf{m} = [m_{ab}] \in C$ such that $\sum_{a,b} m_{ab}$ is minimal and $\mathbf{m}$ is of $\mathfrak{g}_0$-highest weight, that is, $\widetilde{e}_i \mathbf{m} = 0$ for $i \in I_0$.

Case 1. Suppose that $\diamondsuit = (\square\square, \square\square)$. We first note that $\mathbf{m}_{(\overline{n})} \neq [1\ 1]$. Otherwise, $\widetilde{e}_{\overline{n}} \mathbf{m} \neq 0$. Suppose that $\mathbf{m}_{(\overline{n})} = [0\ 1]$. Then there exists $\overline{k+1} \in [\overline{n}]$ such

that $\mathbf{m}_{\overline{k'}} = [0\ 1]$ for $k+1 \leq k' \leq n$ and $\mathbf{m}_{\overline{k'}} = [0\ 0]$ otherwise, since $\widetilde{e}_i \mathbf{m} = 0$ for $i \in I_0$. This implies that $\mathbf{m} = \mathbf{v}_{k,0}$. Suppose that $\mathbf{m}_{(\overline{n})} = [1\ 0]$. Let $\overline{k'+1}$ be the smallest such that $\mathbf{m}_{\overline{k'+1}} = [1\ 0]$. If $\mathbf{m}_{\overline{k'}} = [0\ 0]$, then $\mathbf{m}_{\overline{k'}} = \cdots = \mathbf{m}_{\overline{1}} = [0\ 0]$. If $\mathbf{m}_{\overline{k'}} = [0\ 1]$, then as in the previous case, we have $\mathbf{m}_{\overline{k'}} = \cdots = \mathbf{m}_{\overline{k+1}} = [0\ 1]$ and $\mathbf{m}_{\overline{k}} = \cdots = \mathbf{m}_{\overline{1}} = [0\ 0]$ for some k. This implies that $\mathbf{m} = \mathbf{v}_{k,l}$, where $l = n - k'$, and $C = C(\mathbf{v}_{k,l})$.

Suppose that $C = C(\mathbf{v}_{k'',l''})$ for some k'', l''. We can check that for $\mathbf{m}' = [m'_{ab}] \in C$, $\sum_a m'_{a1} - \sum_a m'_{a2}$ is invariant under $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I$ and $\sum_{a,b} m'_{ab} \geq n - k''$, which implies from the minimality of $\sum_{a,b} m_{ab}$ that $k = k''$ and $l = l''$. This proves the decomposition of $\mathbf{M}$.

Case 2. Suppose that $\diamondsuit = (\square, \square\square)$. As in Case 1, we have $\mathbf{m} = \mathbf{v}_{k,n-k'}$ for some k, k' with $k \leq k' \leq n$. But if $k < k'$, then $\mathbf{m}$ is connected to $\mathbf{v}_{k',0}$ by applying $\widetilde{f}_i$'s for $i \in \{\, k'-1, \ldots, 1, 0 \,\}$, which contradicts the minimality of $\sum_{a,b} m_{ab}$. Hence, $\mathbf{m} = \mathbf{v}_{k,n-k}$, and $C = C(\mathbf{v}_{k,n-k})$. It is also clear that $C(\mathbf{v}_{k,n-k}) = C(\mathbf{v}_{k',n-k'})$ if and only if $k = k'$ for $0 \leq k, k' \leq n$. The proof for $\diamondsuit = (\square\square, \square)$ is almost the same.

Case 3. Suppose that $\diamondsuit = (\begin{smallmatrix}\square\\ \square\end{smallmatrix}, \square\square)$. Then we have $\mathbf{m} = \mathbf{v}_{k,n-k'}$ for some k, k' with $k \leq k' \leq n$. If $k = n$, then $\mathbf{m} = \mathbf{v}_{n,0}$. If $k = 0$, then $\mathbf{m} = \mathbf{v}_{0,n}$ or $\mathbf{v}_{0,n-1}$. If $k \neq 0, n$ and $k < k'$, then $\mathbf{m}$ is connected to $\mathbf{v}_{k',n-k'}$ or $\mathbf{v}_{k'-1,n-k'}$ by applying $\widetilde{f}_i$'s for $i \in \{\, k'-1, \ldots, 1, 0 \,\}$. So, by the minimality of $\sum_{a,b} m_{ab}$, we must have $\mathbf{m} = \mathbf{v}_{k,n-k}$ or $\mathbf{v}_{k,n-k-1}$. On the other hand, it is straightforward to check that

$$\mathrm{S}_w \mathbf{v}_{k,n-k-1} = \mathbf{v}_{k,n-k}, \tag{4.3}$$

where $w \in W$ satisfies $w(\varpi_k) = \varpi_k + \delta$. This implies that $C = C(\mathbf{v}_{k,n-k})$. Finally, if $C = C(\mathbf{v}_{k'',l''})$ for some k'', l'', then we have $k'' = k$ and $l'' = n - k$ from the minimality of $\sum_{a,b} m_{ab}$. Hence we have the decomposition of $\mathbf{M}$. □

4.2. **Decomposition into classical crystals.** Suppose that $\mathfrak{g}$ is of type $\diamondsuit = (\diamondsuit_0, \diamondsuit_n)$ with $\diamondsuit_0$ or $\diamondsuit_n = \square\square$. For a $\mathfrak{g}_0$-dominant weight $\lambda \in P$, let $B_0(\lambda)$ be the crystal of the irreducible $U_q(\mathfrak{g}_0)$-module with highest weight λ.

Theorem 4.2. *For $(k, l) \in H^{\diamondsuit}$, we have the following decomposition of $C(\mathbf{v}_{k,l})$ as a $U_q(\mathfrak{g}_0)$-crystal :*

(1) *If $k = 0$, then $C(\mathbf{v}_{k,l}) \cong B_0(0)$.*

(2) *If* $1 \leq k \leq n$, *then*

$$C(\mathbf{v}_{k,l}) \cong \begin{cases} B_0(\mathrm{cl}(\varpi_k)), & \textit{for } \diamondsuit = (\Box\!\Box, \Box\!\Box), \\ \bigsqcup_{i=0}^{k} B_0(\mathrm{cl}(\varpi_{k-i})), & \textit{for } \diamondsuit = (\Box, \Box\!\Box), \\ B_0(\mathrm{cl}(\varpi_k)), & \textit{for } \diamondsuit = (\Box\!\Box, \Box), \\ \bigsqcup_{i=0}^{[\frac{k}{2}]} B_0(\mathrm{cl}(\varpi_{k-2i}))^{\oplus 2}, & \textit{for } \diamondsuit = (\begin{smallmatrix}\Box\\\Box\end{smallmatrix}, \Box\!\Box) \textit{ with } k \neq n, \\ \bigsqcup_{i=0}^{[\frac{n}{2}]} B_0(\mathrm{cl}(\varpi_{n-2i})), & \textit{for } \diamondsuit = (\begin{smallmatrix}\Box\\\Box\end{smallmatrix}, \Box\!\Box) \textit{ with } k = n, \end{cases}$$

where $B^{\oplus 2} := B \sqcup B$ *for a crystal* B *and* $\varpi_0 := 0$.

Proof. (1) It is clear by (3.4).

(2) Case 1. Suppose that $\diamondsuit = (\Box\!\Box, \Box\!\Box)$. Let $\mathbf{m} \in C(\mathbf{v}_{k,l})$ be given. We may assume that $\widetilde{e}_i \mathbf{m} = 0$ for $i \in I_0$. By the same argument in Proposition 4.1, $\mathbf{m} = \mathbf{v}_{k',l'}$ for some k' and l', which implies that $\mathbf{m} \in C(\mathbf{v}_{k',l'})$. Now from the decomposition of $\mathbf{M}$ as a $U_q'(\mathfrak{g})$-crystal in Proposition 4.1, it follows that $k' = k$ and $l' = l$. Therefore, $C(\mathbf{v}_{k,l})$ is the connected as a $U_q(\mathfrak{g}_0)$-crystal. Since $\mathbf{M}$ is a regular crystal, $C(\mathbf{v}_{k,l})$ is isomorphic to $B_0(\mathrm{cl}(\varpi_k))$ as a $U_q(\mathfrak{g}_0)$-crystal by (4.2). The proof for $\diamondsuit = (\Box\!\Box, \Box)$ is almost the same.

Case 2. Suppose that $\diamondsuit = (\Box, \Box\!\Box)$. Let $\mathbf{m} \in C(\mathbf{v}_{k,n-k})$ be given such that $\widetilde{e}_i \mathbf{m} = 0$ for $i \in I_0$. As in Case 1, we have $\mathbf{m} = \mathbf{v}_{k',l'}$ for some k' and l'. We see that $\mathbf{m} \in C(\mathbf{v}_{n-l',l'})$ by applying $\widetilde{f}_i$'s to $\mathbf{m}$ for $i \in \{n-l'-1, \ldots, 1, 0\}$, and then $l' = n-k$ by Proposition 4.1. Hence $\mathbf{m} = \mathbf{v}_{k',n-k}$ with $0 \leq k' \leq k$. Conversely, for $0 \leq k' \leq k$, we have $\mathbf{v}_{k',n-k} \in C(\mathbf{v}_{k,n-k})$ by applying $\widetilde{f}_i$'s to $\mathbf{v}_{k,n-k}$ for $i \in \{k'-1, \ldots, 1, 0\}$. Hence

$$C(\mathbf{v}_{k,n-k}) = \bigsqcup_{k'=0}^{k} C_0(\mathbf{v}_{k',n-k}), \tag{4.4}$$

where $C_0(\mathbf{m})$ denotes the connected component of $\mathbf{m}$ as a $U_q(\mathfrak{g}_0)$-crystal. Finally, $C_0(\mathbf{v}_{k',n-k})$ $(k \neq 0)$ is isomorphic to $B_0(\mathrm{cl}(\varpi_{k'}))$ by (4.2), while $\mathbf{v}_{0,n}$ gives the trivial crystal $B_0(0)$. This proves the decomposition of $C(\mathbf{v}_{k,n-k})$.

Case 3. Suppose that $\diamondsuit = (\begin{smallmatrix}\Box\\\Box\end{smallmatrix}, \Box\!\Box)$. Similarly, from the argument in the proof of Proposition 4.1, we see that if $k \neq n$, then

$$C(\mathbf{v}_{k,n-k}) = \bigsqcup_{i=0}^{[\frac{k}{2}]} \left(C_0(\mathbf{v}_{k-2i,n-k}) \sqcup C_0(\mathbf{v}_{k-2i,n-k-1}) \right), \tag{4.5}$$

where both $C_0(\mathbf{v}_{k-2i,n-k})$ are $C_0(\mathbf{v}_{k-2i,n-k-1})$ are isomorphic to $B_0(\mathrm{cl}(\varpi_{k-2i}))$ for $k-2i\neq 0$, and $C_0(\mathbf{v}_{0,n-k-1})\cong C_0(\mathbf{v}_{0,n-k})\cong B_0(0)$. Also, if $k=n$, then we have

$$C(\mathbf{v}_{n,0})=\bigsqcup_{i=0}^{[\frac{n}{2}]} C_0(\mathbf{v}_{n-2i,0}), \tag{4.6}$$

where $C_0(\mathbf{v}_{n-2i,0})\cong B_0(\mathrm{cl}(\varpi_{n-2i}))$. The proof completes. $\square$

4.3. **Order 2 symmetry on $A^{(2)}_{2n-1}$-crystals.** Let us consider a symmetry of the crystal $C(\mathbf{v}_{k,n-k})$ for $\lozenge = (\begin{smallmatrix}\square\\ \square\end{smallmatrix}, \square\square)$ and $k\neq n$, which will be necessary in the next section.

Let $\mathbf{m}=[m_{ab}]\in C(\mathbf{v}_{k,n-k})$ be given. For each $a\in[\overline{n}]$, we may regard $\mathbf{m}_{(a)}$ as a crystal element over $U_q(\mathfrak{sl}_2)$ with Kashiwara operators $\widetilde{E}$ and $\widetilde{F}$ such that $\widetilde{F}[1\ 0]=[0\ 1]$, $\widetilde{E}[0\ 1]=[1\ 0]$, and $\widetilde{X}[0\ 0]=\widetilde{X}[1\ 1]=0$ $(X=E,F)$. Then we understand $\mathbf{m}$ as $\mathbf{m}_{(\overline{1})}\otimes\cdots\otimes\mathbf{m}_{(\overline{n})}$ by the tensor product rule. Put

$$\sigma(\mathbf{m})=(\varepsilon(\mathbf{m}),\varphi(\mathbf{m})), \tag{4.7}$$

where $\varepsilon(\mathbf{m})=\max\{\,k\,|\,\widetilde{E}^k\mathbf{m}\neq 0\,\}$ and $\varphi(\mathbf{m})=\max\{\,k\,|\,\widetilde{F}^k\mathbf{m}\neq 0\,\}$.

Lemma 4.3. *Under the above hypothesis, we have $\sigma(\mathbf{m})=(2i,n-k)$ or $(2i+1,n-k-1)$ for some $i\in\mathbb{Z}_{\geq 0}$.*

Proof. Note that $\sigma(\mathbf{v}_{k-2i,n-k})=(2i,n-k)$ and $\sigma(\mathbf{v}_{k-2i,n-k-1})=(2i+1,n-k-1)$ for $0\leq i\leq[\frac{k}{2}]$. Since $\mathbf{m}$ can be viewed as an element in a $(U_q(\mathfrak{sl}_{n-1}),U_q(\mathfrak{sl}_2))$-bicrystal $\mathbf{M}$ (cf. [5]) and $\widetilde{e}_n$, $\widetilde{f}_n$ are either a trivial or isomorphism of $U_q(\mathfrak{sl}_2)$-crystals, σ is constant on each connected component in $C(\mathbf{v}_{k,n-k})$ as a $U_q(\mathfrak{g}_0)$-crystal. Hence the claim follows from (4.5). $\square$

Let

$$\begin{aligned} C(\mathbf{v}_{k,n-k})^+ &=\{\,\mathbf{m}\in C(\mathbf{v}_{k,n-k})\,|\,\varphi(\mathbf{m})=n-k\,\},\\ C(\mathbf{v}_{k,n-k})^- &=\{\,\mathbf{m}\in C(\mathbf{v}_{k,n-k})\,|\,\varphi(\mathbf{m})=n-k-1\,\}.\end{aligned} \tag{4.8}$$

By Lemma 4.3, $C(\mathbf{v}_{k,n-k})=C(\mathbf{v}_{k,n-k})^+\sqcup C(\mathbf{v}_{k,n-k})^-$. Now, define a map

$$\varsigma: C(\mathbf{v}_{k,n-k})\longrightarrow C(\mathbf{v}_{k,n-k}) \tag{4.9}$$

by $\varsigma(\mathbf{m})=\widetilde{F}\mathbf{m}$ (resp. $\widetilde{E}\mathbf{m}$) if $\mathbf{m}\in C(\mathbf{v}_{k,n-k})^+$ (resp. $C(\mathbf{v}_{k,n-k})^-$). By definition, $\varsigma^2(\mathbf{m})=\mathbf{m}$ for all $\mathbf{m}$.

Example 4.4. Let

$$\mathbf{m}=\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}\in C(\mathbf{v}_{1,1}).$$

Then $\sigma(\mathbf{m}) = (1,1)$ and $\mathbf{m} \in C(\mathbf{v}_{1,1})^-$. Hence

$$\varsigma(\mathbf{m}) = \widetilde{E}\mathbf{m} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \in C(\mathbf{v}_{1,1})^+.$$

Proposition 4.5. *ς commutes with $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I$. Hence ς is an isomorphism of $U_q'(\mathfrak{g})$-crystals.*

Proof. It is straightforward to check that ς commutes with $\widetilde{e}_0$ and $\widetilde{f}_0$. Also, it follows from the argument in the proof of Lemma 4.3 that ς is an isomorphism of $U_q(\mathfrak{g}_0)$-crystals. This proves the claim. $\square$

We denote by $C(\mathbf{v}_{k,n-k})/\langle\varsigma\rangle$ the set of orbits in $C(\mathbf{v}_{k,n-k})$ under the action of the automorphism group $\{1, \varsigma\}$ with a graph structure induced from $C(\mathbf{v}_{k,n-k})$ (see Figure 3).

5. Realization of level zero fundamental representations

5.1. $W(\varpi_n)$ **or** $W(\varpi_{n-1})$ **of type** $B_n^{(1)}$**,** $D_n^{(1)}$**,** $D_{n+1}^{(2)}$**.** Suppose that $\mathfrak{g}$ is of type $\diamondsuit = (\diamondsuit_0, \diamondsuit_n)$ with $\diamondsuit_0$, $\diamondsuit_n \neq \square\square$, and $\mathbf{v}_k$ is a $U_q(\mathfrak{g}_0)$-highest weight element in $\mathbf{B}$ $(k = n, n-1)$ (see (4.1)). As a $U_q(\mathfrak{g}_0)$-crystal, $C(\mathbf{v}_k)$ is isomorphic to $B_0(\mathrm{cl}(\varpi_k))$. Let v_k $(k = n, n-1)$ be a $U_q(\mathfrak{g}_0)$-highest weight vector in $\Lambda(V)$ such that $v_k \equiv \mathbf{v}_k$ (mod $q_s\mathscr{L}((\mathscr{V}))$). Let

$$W_k = U_q'(\mathfrak{g})v_k. \tag{5.1}$$

By (4.1), $(\mathscr{L}(\Lambda(V)) \cap W_k, C(\mathbf{v}_k))$ is a crystal base of W_k and

$$\Lambda(V) = \begin{cases} W_n, & \text{if } \diamondsuit = (\square, \square), \\ W_n \oplus W_{n-1}, & \text{otherwise.} \end{cases} \tag{5.2}$$

In particular, the weights of W_k is contained in the convex hull of $W\mathrm{cl}(\varpi_k)$, which implies that v_k is an extremal weight vector by [13, Theorem 5.3]. Since $\dim(W_k)_{\mathrm{cl}(\varpi_k)} = 1$ and $C(\mathbf{v}_k)$ is connected, W_k is irreducible and therefore $W_k \cong W(\varpi_k)_{a_k}$ for some $a_k \in K\setminus\{0\}$ by Theorem 2.2 (6). Choose $w \in W$ such that $w(\varpi_k) = \varpi_k + \delta$. Then by Propositions 3.1 and 3.2, it is easy to check that $S_w v_k = v_k$. Since $S_w u_{\varpi_k} = z_k u_{\varpi_k}$ in $V(\varpi_k)$ and $W(\varpi_k)_a = V(\varpi_k)/(z_k - a)V(\varpi_k)$, we have $a_k = 1$. Therefore

$$W_k \cong W(\varpi_k). \tag{5.3}$$

We remark that the construction of $W(\varpi_k)$ $(k = n-1, n)$ is already well-known. The decomposition of $\Lambda(V)^{\otimes 2}$ follows from (5.2), (5.3) and simplicity of tensor product of $W(\varpi_k)$'s [13, Theorem 9.2].

5.2. $W(\varpi_k)$ **$(1 \leq k \leq n)$ of type $C_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n}^{(2)\dagger}$, $A_{2n-1}^{(2)}$.** Suppose that $\mathfrak{g}$ is of type $\Diamond = (\diamondsuit_0, \diamondsuit_n)$ with $\diamondsuit_0$ or $\diamondsuit_n = \square\square$. For a $U_q(\mathfrak{g}_0)$-highest weight crystal element $\mathbf{v}_{k,l} \in \mathbf{M}$, let $v_{k,l}$ be an associated $U_q(\mathfrak{g}_0)$-highest weight vector in $\Lambda(V)^{\otimes 2}$ such that $v_{k,l} \equiv \mathbf{v}_{k,l} \pmod{q\mathscr{L}(\Lambda(V)^{\otimes 2})}$. For $(k,l) \in H^{\Diamond}$, put

$$W_{k,l} = U_q'(\mathfrak{g})v_{k,l}. \tag{5.4}$$

For $\Diamond = (\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square)$ with $k \neq 0, n$, put

$$W_{k,n-k}^{\pm} = U_q'(\mathfrak{g})(v_{k,n-k} \pm v_{k,n-k-1}). \tag{5.5}$$

Proposition 5.1. *We have*

$$\Lambda(V)^{\otimes 2} = \bigoplus_{(k,l) \in H^{\Diamond}} W_{k,l},$$

where $W_{k,l}$ has a polarizable crystal base $(L(W_{k,l}), B(W_{k,l}))$ with $L(W_{k,l}) = L(\Lambda(V)^{\otimes 2}) \cap W_{k,l}$ and $B(W_{k,l}) = C(\mathbf{v}_{k,l})$.

Proof. Note that $L(W_{k,l})$ is invariant under $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I$, and $C(\mathbf{v}_{k,l})$ is linearly independent subset of $L(W_{k,l})/q_s L(W_{k,l})$. By Proposition 4.1, $L(\Lambda(V)^{\otimes 2}) = \bigoplus_{(k,l)\in H^{\Diamond}} L(W_{k,l})$, which implies that $\Lambda(V)^{\otimes 2} = \bigoplus_{(k,l)\in H^{\Diamond}} W_{k,l}$, and $(L(W_{k,l}), C(\mathbf{v}_{k,l}))$ is a crystal base of $W_{k,l}$. The polarizability follows from that of $\Lambda(V)^{\otimes 2}$. $\Box$

Lemma 5.2. *For $(k,l) \in H^{\Diamond}$ with $k \neq 0$, $v_{k,l}$ is an extremal weight vector of weight* $\mathrm{cl}(\varpi_k)$.

Proof. We have shown in Proposition 5.1 that $W_{k,l}$ has a crystal base with crystal $C(\mathbf{v}_{k,l})$. By Theorem 4.2 (2), we see that the weights of $W_{k,l}$ or $C(\mathbf{v}_{k,l})$ belong to the convex hull of $W\mathrm{cl}(\varpi_k)$. Then by [13, Theorem 5.3], $v_{k,l}$ is an extremal weight vector. $\Box$

Lemma 5.3. *For $(k,l) \in H^{\Diamond}$ with $k \neq 0$ and $w \in W$,*

$$S_w v_{k,l} = \mathtt{S}_w \mathbf{v}_{k,l} + \sum_{\substack{\mathbf{m} \in \mathbf{M} \setminus \{\mathbf{v}_{k,l}\} \\ \mathrm{wt}(\mathbf{m}) = w(\mathrm{cl}(\varpi_k))}} a_{\mathbf{m}} \mathbf{m},$$

where $a_{\mathbf{m}} \in q_s\mathbb{Q}[q_s]$.

Proof. It follows directly from the definition of e_i and f_i on $\Lambda(V)^{\otimes 2}$ and the induction on the length of w. $\Box$

Theorem 5.4. *Let $(k,l) \in H^{\Diamond}$ be given.*

(1) *If $\Diamond \neq (\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square)$ or $\Diamond = (\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square)$ with $k = 0, n$, then*

$$W_{k,l} \cong W(\varpi_k).$$

(2) *If* $\Diamond = (\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square)$ *and* $k \neq 0, n$*, then* $W_{k,n-k} = W^+_{k,n-k} \oplus W^-_{k,n-k}$ *and*

$$W^{\pm}_{k,n-k} \cong W(\varpi_k)_{\pm 1}$$

Here we assume that $W(\varpi_0) = W(0)$ *is the trivial* $U_q'(\mathfrak{g})$*-module of one dimension.*

Proof. Case 1. Suppose that either $\Diamond \neq (\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square)$ or $\Diamond = (\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square)$ with $k = 0, n$. Recall that $B(W_{k,l})$ is connected and $\dim(W_{k,l})_{\mathrm{cl}(\varpi_k)} = \dim C(\mathbf{v}_{k,l})_{\mathrm{cl}(\varpi_k)} = 1$ by Theorem 4.2. Hence $W_{k,l}$ is an irreducible $U_q'(\mathfrak{g})$-module generated by $v_{k,l}$. Assume that $k \neq 0$. By Lemma 5.2 and Theorem 2.2 (6), $W_{k,l} \cong W(\varpi_k)_{a_{k,l}}$ for some $a_{k,l} \in K \setminus \{0\}$. Choose $w \in W$ such that $w(\varpi_k) = \varpi_k + d_k\delta$. Since $\dim C(\mathbf{v}_{k,l})_{\mathrm{cl}(\varpi_k)} = 1$, we have $\mathtt{S}_w \mathbf{v}_{k,l} = \mathbf{v}_{k,l}$. By using Lemma 5.3, we have

$$S_w v_{k,l} = \mathtt{S}_w \mathbf{v}_{k,l} + \sum_{\substack{\mathbf{m} \in \mathbf{M} \setminus \{\mathbf{v}_{k,l}\} \\ \mathrm{wt}(\mathbf{m}) = \mathrm{cl}(\varpi_k)}} a_{\mathbf{m}} \mathbf{m} = \mathbf{v}_{k,l} + \sum_{\substack{\mathbf{m} \in \mathbf{M} \setminus \{\mathbf{v}_{k,l}\} \\ \mathrm{wt}(\mathbf{m}) = \mathrm{cl}(\varpi_k)}} a_{\mathbf{m}} \mathbf{m}.$$

Since $S_w v_{k,l} = a_{k,l} v_{k,l}$, we have $a_{k,l} = 1$, and hence $W_{k,l}$ is isomorphic to $W(\varpi_k)$.

Case 2. Suppose that $\Diamond = (\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square)$ and $k \neq 0, n$. Put

$$\begin{aligned} L(W^{\pm}_{k,n-k}) &= L(W_{k,n-k}) \cap W^{\pm}_{k,n-k}, \\ B(W^{\pm}_{k,n-k}) &= \{\, \mathbf{m} \pm \varsigma(\mathbf{m}) \,|\, \mathbf{m} \in C(\mathbf{v}_{k,n-k})^+ \,\}. \end{aligned}$$

Note that $B(W^+_{k,n-k}) \sqcup B(W^-_{k,n-k})$ is a $\mathbb{Q}$-basis of $L(W_{k,n-k})/qL(W_{k,n-k})$ since it is orthogonal. By Proposition 4.5, $B(W^{\pm}_{k,n-k})$ is a linearly independent subset of $L(W^{\pm}_{k,n-k})/q_s L(W^{\pm}_{k,n-k})$, which is invariant under $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I$ up to scalar multiplication by ± 1. This implies that

$$W_{k,n-k} = W^+_{k,n-k} \oplus W^-_{k,n-k},$$

and $(L(W^{\pm}_{k,n-k}), B(W^{\pm}_{k,n-k}))$ is a pseudo crystal base of $W^{\pm}_{k,n-k}$ (cf. [11]).

Since $\dim(W^{\pm}_{k,n-k})_{\mathrm{cl}(\varpi_k)} = 1$, by the same arguments as in Case 1 we have $W^{\pm}_{k,n-k} \cong W(\varpi_k)_{a^{\pm}_{k,n-k}}$ for some $a^{\pm}_{k,n-k} \in K \setminus \{0\}$.

Finally, let $w \in W$ be given such that $w(\varpi_k) = \varpi_k + \delta$. Then we can check that $\mathtt{S}_w \mathbf{v}_{k,n-k} = \mathbf{v}_{k,n-k-1}$ and $\mathtt{S}_w \mathbf{v}_{k,n-k-1} = \mathbf{v}_{k,n-k}$. Since $\mathtt{S}_w(\mathbf{v}_{k,n-k} \pm \mathbf{v}_{k,n-k-1}) = \pm(\mathbf{v}_{k,n-k} \pm \mathbf{v}_{k,n-k-1})$, it follows from the same argument as in Case 1 that $S_w v^{\pm}_{k,n-k} = \pm v^{\pm}_{k,n-k}$ or $a^{\pm}_{k,n-k} = \pm 1$. Therefore, $W^{\pm}_{k,n-k}$ is isomorphic to $W(\varpi_k)_{\pm 1}$. $\square$

Let $B(W(\varpi_k))$ denote the crystal of $W(\varpi_k)$ for $k \in I_0$.

Corollary 5.5. *For* $(k,l) \in H^{\Diamond}$ *with* $k \neq 0$*, we have*

$$B(W(\varpi_k)) \cong \begin{cases} C(\mathbf{v}_{k,n-k})/\langle \varsigma \rangle, & \text{if } \Diamond = (\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square) \text{ and } k \neq n, \\ C(\mathbf{v}_{k,l}), & \text{otherwise.} \end{cases}$$

Corollary 5.6. *As a $U_q'(\mathfrak{g})$-module, $\Lambda(V)^{\otimes 2}$ is isomorphic to*

$$\begin{cases} \bigoplus_{0\leq k\leq n} W(\varpi_k)^{\oplus n-k+1}, & \text{if } \diamondsuit = (\square\square, \square\square), \\ \bigoplus_{0\leq k\leq n} W(\varpi_k), & \text{if } \diamondsuit = (\square, \square\square)\ ,\ (\square\square, \square), \\ W(\varpi_n) \oplus W(0)^{\oplus 2} \oplus \bigoplus_{\substack{1\leq k\leq n-1 \\ a=\pm 1}} W(\varpi_k)_a, & \text{if } \diamondsuit = (\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square). \end{cases}$$

Remark 5.7. By Theorem 4.2, we recover the decomposition of $B(W(\varpi_k))$ into $U_q(\mathfrak{g}_0)$-crystals (see [8] for a more general case of $\mathfrak{g}$ and $W(\varpi_k)$). For example,

$$C(\mathbf{v}_{k,n-k})/\langle\varsigma\rangle \cong \bigsqcup_{i=0}^{[\frac{k}{2}]} B_0(\mathrm{cl}(\varpi_{k-2i})),$$

as a $U_q(\mathfrak{g}_0)$-crystal.

5.3. **Description of $B(W(\varpi_k))$.** For $\mathbf{m} = [m_{ab}] \in \mathbf{M}$, we define

$$\sigma(\mathbf{m}) = (\varepsilon(\mathbf{m}), \varphi(\mathbf{m})),$$

as in (4.7). For $k \in \mathbb{Z}$, put $\langle k\rangle = \max\{k, 0\}$. By [14, Proposition 2.1.1], we have

$$\varepsilon(\mathbf{m}) = \max\left\{ \sum_{1\leq i\leq k} \langle m_{\bar{i}2} - m_{\bar{i}1}\rangle - \sum_{1\leq i<k} \langle m_{\bar{i}1} - m_{\bar{i}2}\rangle \,\middle|\, 1\leq k\leq n \right\},$$

$$\varphi(\mathbf{m}) = \max\left\{ \sum_{k\leq i<n} \left\{ \langle m_{\bar{i}1} - m_{\bar{i}2}\rangle - \langle m_{\overline{i+1}2} - m_{\overline{i+1}1}\rangle \right\} + \langle m_{\bar{n}1} - m_{\bar{n}2}\rangle \,\middle|\, 1\leq k\leq n \right\}.$$

By using the result [16] on the crystal of type B_n and C_n, we have the following characterization of $B(W(\varpi_k))$ $(1\leq k\leq n)$ of type $C_n^{(1)}$, $A_{2n}^{(2)}$, $A_{2n}^{(2)\dagger}$, $A_{2n-1}^{(2)}$ in terms of binary matrices $\mathbf{m}$ in $\mathbf{M}$ with constraints on $\sigma(\mathbf{m})$.

Theorem 5.8. *For $1\leq k\leq n$, we have the following.*

(1) *If $\diamondsuit = (\square\square, \square\square)$ or $\mathfrak{g} = C_n^{(1)}$, then*

$$B(W(\varpi_k)) = \{\, \mathbf{m} \,|\, \sigma(\mathbf{m}) = (n-k, 0)\,\}.$$

(2) *If $\diamondsuit = (\square, \square\square)$ or $\mathfrak{g} = A_{2n}^{(2)}$, then*

$$B(W(\varpi_k)) = \{\, \mathbf{m} \,|\, \sigma(\mathbf{m}) = (k-l, n-k) \text{ for } 0\leq l\leq k\,\}.$$

(3) *If $\diamondsuit = (\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square)$ or $\mathfrak{g} = A_{2n-1}^{(2)}$, then*

$$B(W(\varpi_k)) = \begin{cases} \{\, \mathbf{m} \,|\, \sigma(\mathbf{m}) = (2l, 0) \text{ for } 0\leq l\leq [n/2]\,\}, & (k=n), \\ \{\, \mathbf{m} + \varsigma(\mathbf{m}) \,|\, \sigma(\mathbf{m}) = (2l, n-k) \text{ for } 0\leq l\leq [n/2]\,\}, & (k\neq n). \end{cases}$$

(4) *If* $\diamondsuit = (\square\square, \square)$ *or* $\mathfrak{g} = A_{2n}^{(2)\dagger}$, *then*

$$B(W(\varpi_k)) = \left\{ \mathbf{m} \;\middle|\; \begin{array}{l} (1)\ \sum_a m_{a2} = n-k+s \textit{ for some } s \geq 0, \\ (2)\ \sum_a m_{a1} = t+s \textit{ for some } t \geq 0, \\ (3)\ \sigma(\mathbf{m}) = (n-k-p, t-p) \textit{ for some } 0 \leq p \leq \min\{t, n-k\}. \end{array} \right\}.$$

Proof. For $\mathbf{m} \in \mathbf{M}$ and $b = 1, 2$, put $|\mathbf{m}^{(b)}| = \sum_{ab} m_{ab}$.

(1) Suppose that $\diamondsuit = (\square\square, \square\square)$ or $C_n^{(1)}$.

We have $B(W(\varpi_k)) \cong C(\mathbf{v}_{k,0}) = C_0(\mathbf{v}_{k,0})$ by Corollary 5.5 and Theorem 4.2. For $\mathbf{m} \in C(\mathbf{v}_{k,0})$, we have $|\mathbf{m}^{(2)}| - |\mathbf{m}^{(1)}| = n-k$ and $\sigma(\mathbf{m}) = (n-k, 0)$ since σ is invariant under $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I_0$.

Conversely, let C be the set of $\mathbf{m} \in \mathbf{M}$ such that $\sigma(\mathbf{m}) = (n-k, 0)$. By the signature rule of tensor product of crystals (cf. [14, Remark 2.1.2]), we see that $|\mathbf{m}^{(2)}| - |\mathbf{m}^{(1)}| = n-k$. For $\mathbf{m} \in C$, we identify each column $\mathbf{m}^{(b)}$ ($b = 1, 2$) with a single column semistandard tableau $T^{(b)}$ of length $|\mathbf{m}^{(b)}|$ with entries in $[\overline{n}]$ such that $a \in [\overline{n}]$ appears in $T^{(b)}$ if and only if $m_{ab} = 1$. Let T be a tableau of shape $(2^{|\mathbf{m}^{(1)}|}, 1^{|\mathbf{m}^{(2)}|-|\mathbf{m}^{(1)}|})$ whose left (resp. right) column is $T^{(2)}$ (resp. $T^{(1)}$). Then T is semistandard by [16, Lemma 6.2]. Since the actions of $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I_0$ on T coincides with those on $\mathbf{m}$ (see [16, Section 7.1]), the map $\mathbf{m} \mapsto T$ gives a $U_q(\mathfrak{g}_0)$-crystal (or $U_q(C_n)$-crystal) isomorphism from C to the set of semistandard tableaux of shape $(2^s, 1^{n-k})$ ($0 \leq s \leq k$) with entries in $[\overline{n}]$, which is isomorphic to $B_0(\mathrm{cl}(\varpi_k))$ by [16, Theorem 7.1]. This implies that $C = C(\mathbf{v}_{k,0})$. Hence, we have $B(W(\varpi_k)) = \{\, \mathbf{m} \,|\, \sigma(\mathbf{m}) = (n-k, 0) \,\}$.

(2) Suppose that $\diamondsuit = (\square, \square\square)$ or $A_{2n}^{(2)}$.

By (4.4), we have $B(W(\varpi_k)) \cong \bigsqcup_{l=0}^{k} C_0(\mathbf{v}_{l,n-k})$ as a $U_q(C_n)$-crystal. Since $\widetilde{F}^{n-k}\mathbf{m} \in C(\mathbf{v}_{l,0})$ for $\mathbf{m} \in C(\mathbf{v}_{l,n-k})$ and $\widetilde{F}$ commutes with $\widetilde{e}_i$ and $\widetilde{f}_i$ for $i \in I_0$, $\widetilde{F}^{n-k} : C(\mathbf{v}_{l,n-k}) \longrightarrow C(\mathbf{v}_{l,0})$ is a $U_q(C_n)$-crystal isomorphism, which implies that $C(\mathbf{v}_{l,n-k}) = \{\, \mathbf{m} \,|\, \sigma(\mathbf{m}) = (k-l, n-k) \,\}$. Hence, we have $B(W(\varpi_k)) = \{\, \mathbf{m} \,|\, \sigma(\mathbf{m}) = (k-l, n-k)$ for $0 \leq l \leq k \,\}$.

(3) Suppose that $\diamondsuit = (\begin{smallmatrix}\square\\\square\end{smallmatrix}, \square\square)$ or $A_{2n-1}^{(2)}$.

If $k = n$, then by (4.6), $B(W(\varpi_n)) \cong \bigsqcup_{l=0}^{[\frac{n}{2}]} C_0(\mathbf{v}_{n-2l,0})$ as a $U_q(C_n)$-crystal. Similar to the above cases, we have $B(W(\varpi_n)) = \{\, \mathbf{m} \,|\, \sigma(\mathbf{m}) = (2l, 0)$ for $0 \leq l \leq [n/2] \,\}$.

If $k \neq n$, then by (4.5), we have $B(W(\varpi_k)) = \{\, \mathbf{m} + \varsigma(\mathbf{m}) \,|\, \sigma(\mathbf{m}) = (2l, n-k)$ for $0 \leq l \leq [n/2] \,\}$.

(4) Suppose that $\diamondsuit = (\square\square, \square)$ or $A_{2n}^{(2)\dagger}$.

We have $B(W(\varpi_k)) \cong C(\mathbf{v}_{k,0}) = C_0(\mathbf{v}_{k,0})$. It is not difficult to see that for $\mathbf{m} \in C(\mathbf{v}_{k,0})$, $|\mathbf{m}^{(2)}| = n-k+s$, $|\mathbf{m}^{(1)}| = s+t$ for some $s, t \geq 0$ and $\sigma(\mathbf{m}) = (n-k-p, t-p)$ for some $0 \leq p \leq \min(t, n-k)$.

Conversely, let C be the set of $\mathbf{m}$ such that $|\mathbf{m}^{(2)}| = n-k+s$, $|\mathbf{m}^{(1)}| = s+t$ for some $s, t \geq 0$ and $\sigma(\mathbf{m}) = (n-k-p, t-p)$ for some $0 \leq p \leq \min(t, n-k)$. Let $T^{(b)}$ be the semistandard tableau of single column associated to $\mathbf{m}^{(b)}$ for $b = 1, 2$. Let T be a tableau of skew shape $(2^{s+t}, 1^{n-k})/(1^t)$, whose left (resp. right) column is $T^{(2)}$ (resp. $T^{(1)}$). By [16, Lemma 6.2], the map $\mathbf{m} \mapsto T$ gives a $U_q(\mathfrak{g}_0)$-crystal (or $U_q(B_n)$-crystal) isomorphism from C to the set of semistandard tableaux of shape $(2^{s+t}, 1^{n-k})/(1^t)$ $(s, t \geq 0,\ 0 \leq s \leq k)$ with entries in $[\overline{n}]$, which is isomorphic to $B_0(\mathrm{cl}(\varpi_k))$ [16, Theorem 7.1]. This implies that $C = C(\mathbf{v}_{k,0}) \cong B(W(\varpi_k))$. $\square$

Remark 5.9. By using the correspondence between $\mathbf{m} \in \mathbf{M}$ and a semistandard tableaux of two-column skew shape together with [16, Section 7.2], we also obtain an explicit bijection from a classical crystal $C_0(\mathbf{v}_{k,0})$ of type B_n or C_n to the that of Kashiwara-Nakashima tableaux of (non-spinor) single column with length k.

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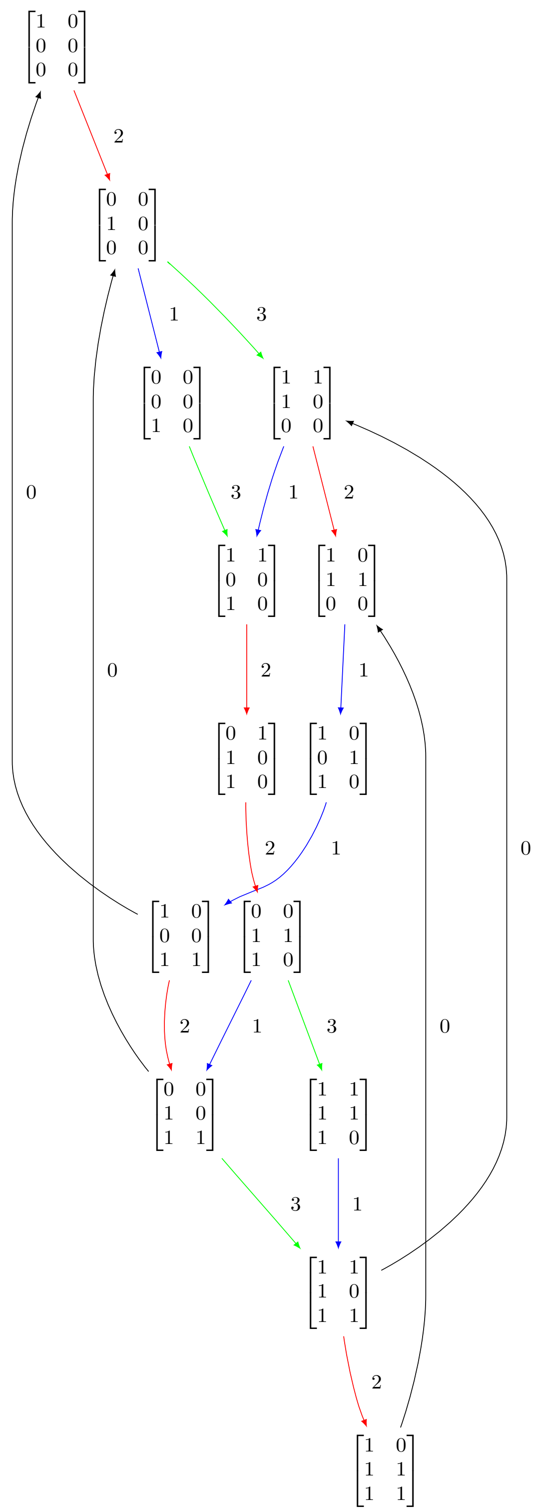

FIGURE 1. $B(W(\varpi_2)) = C(\mathbf{v}_{2,1})$ of type $C_3^{(1)}$

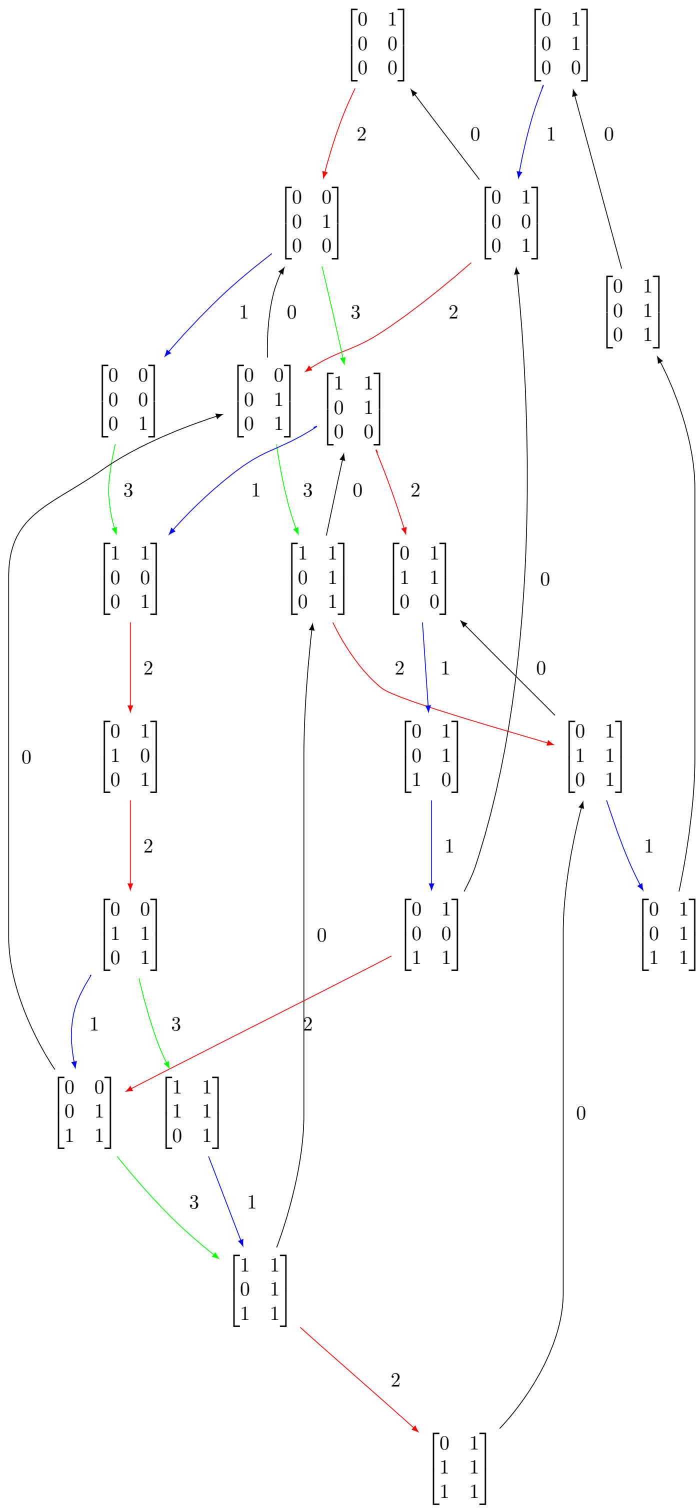

Figure 2. $B(W(\varpi_2)) = C(\mathbf{v}_{2,0})$ of type $A_6^{(2)}$

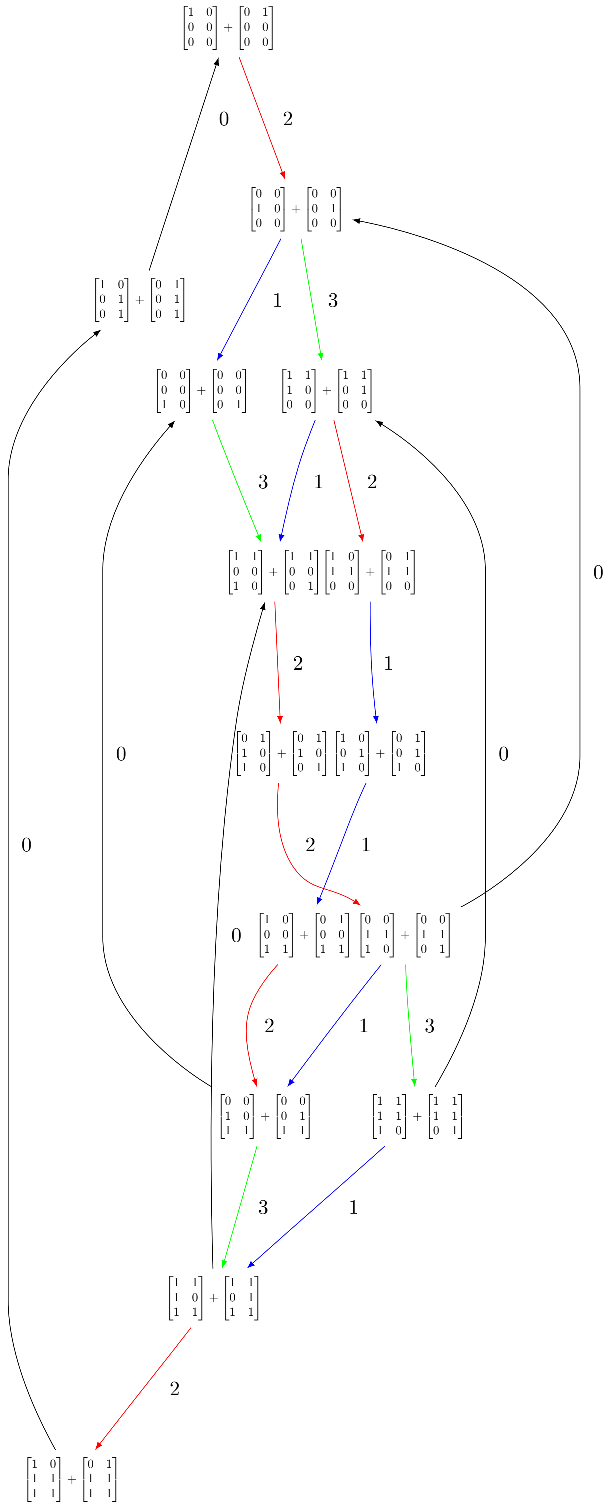

FIGURE 3. $B(W(\varpi_2))$ of type $A_5^{(2)}$ with vertices $\mathbf{m} + \varsigma(\mathbf{m})$ for $\mathbf{m} \in C(\mathbf{v}_{2,1})^+$